

POSITIVE MASS THEOREM FOR THE PANEITZ-BRANSON OPERATOR

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ABSTRACT. We prove that under suitable assumptions, the constant term in the Green function of the Paneitz-Branson operator on a compact Riemannian manifold (M, g) is positive unless (M, g) is conformally diffeomorphic to the standard sphere. The proof is inspired by the positive mass theorem on spin manifolds by Ammann-Humbert [AH03].

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. We denote by Q_g the Q -curvature for the metric g defined by

$$Q_g := \frac{n^2 - 4}{8n(n-1)^2} S_g^2 - \frac{2}{(n-2)^2} |E_g|^2 + \frac{1}{2(n-1)} \Delta_g S_g,$$

where $\Delta_g = -\operatorname{div}_g \nabla$ is the Laplace-Beltrami operator, S_g stands for the scalar curvature of g , $|E_g|$ denotes the g -norm of the Einstein tensor $E_g := \operatorname{Ric}_g - \frac{S_g}{n} g$ and Ric_g is the Ricci curvature of g . The Paneitz-Branson operator introduced for $n = 4$ by Paneitz in [Pa83] and whose definition was generalized in dimension greater than 5 by Branson [Br87], is defined for all $u \in C^\infty(M)$ by

$$P_g u := \Delta_g^2 u - \operatorname{div}_g (A_g du) + \frac{n-4}{2} Q_g u$$

where

$$A_g := \frac{(n-2)^2 + 4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} \operatorname{Ric}_g.$$

This operator is closely related to the problem of prescribing Q -curvature in a conformal class as well as the Yamabe operator (see (5) below) is related to the problem of prescribing the scalar curvature in a conformal class. It is a conformally covariant operator in the sense that if $g' = e^{2f} g$ is conformal to g , then for all $v \in C^\infty(M)$,

$$P_{g'}(e^{-\frac{n-4}{2}f} v) = e^{-\frac{n+4}{2}f} P_g(v).$$

In particular, if $n \geq 5$, and if we set $u = e^{\frac{n-4}{2}f}$ so that $g' = u^{\frac{4}{n-4}} g$, we get for all $v \in C^\infty(M)$

$$P_{g'}(u^{-1}v) = u^{-\frac{n+4}{n-4}} P_g(v). \quad (1)$$

From now on, we make the following assumptions:

- (a) g is conformally flat;
- (b) $n \geq 5$;

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- (c) the Yamabe invariant is positive (see for instance [Au98] or [He97]) i.e. g is conformal to a metric g' for which the scalar curvature is positive.
 (d) the operator P_g is positive.

Under Assumptions (a) to (d), it is well known that the Green's function G_g of P_g exists, is unique and smooth on $M \setminus \{p\}$. By the conformal covariance of the Paneitz-Branson operator, if $g' = u^{\frac{4}{n-4}}g$ is conformal to g , then

$$G_{g'}(x, y) = \frac{G_g(x, y)}{u(x)u(y)}.$$

Now, let $p \in M$. By (1), up to a conformal change of metric, we can assume

- (a') g is flat around p .

Then, it is known that we have the following expansion when x is close to p ,

$$G_g(x, p) = \frac{1}{2(n-2)(n-4)\omega_{n-1}d_g(x, p)^{n-4}} + A + \alpha_p(x) \quad (2)$$

where ω_{n-1} stands for the volume of the $(n-1)$ -dimensional sphere, $A \in \mathbb{R}$, α_p is a smooth function defined around p and satisfying $\alpha_p(p) = 0$. By analogy to the case of the conformal Laplacian (see again [Au98, He97]), the number A is called the *mass of the Paneitz-Branson operator*. If $g' = u^{\frac{4}{n-4}}g$ is another metric conformal to g and flat around p , then the mass A' corresponding to the metric g' is given by

$$A' = \frac{A}{u(p)^2}.$$

Hence, the mass A depends on the choice of the metric in the conformal class, but not its sign. This is the reason why in the statement of Theorem 1.1 below, we do not need to assume (a').

We also make the following assumption

- (e) $G_g > 0$ on $M \setminus \{p\}$.

For interesting results concerning Assumptions (d) and (e), the reader may refer to Grunau-Robert [GR07].

The main result of the paper is the following:

Theorem 1.1. *Under assumptions (a) to (e), the mass A satisfies*

$$A \geq 0$$

with equality if and only if (M, g) is conformally diffeomorphic to the sphere.

Theorem 1.1 has been already proven with the additional assumption that the Poincaré exponent is small enough (see [QR06a, QR06b]). In this case, Qing and Raske proved also the positivity of the Green's function of G_g .

Our proof is inspired from the positive mass theorem on spin manifolds by Ammann-Humbert in [AH03] (see also Raulot [Ra07]). The difficulty here is to overcome

the fact that on non-spin manifolds, there is no equivalent of the Schrödinger-Lichnerowicz Formula.

Hebey and Robert proved the nice following result which is an analogue for geometric equations of order 4 of a hard problem concerning the Yamabe Equation:

Theorem (Hebey, Robert; [HR04]). *Let (M, g) be a conformally flat compact manifold of dimension $n \geq 5$. Assume g has a positive Yamabe invariant, that P_g is positive as well as its Green function and that the mass of P_g is positive. Then, the geometric equation*

$$P_g u = u^{\frac{n+4}{n-4}}$$

is compact.

In particular, together with Theorem 1.1, we get rid of the positivity of the mass.

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2. PROOF OF THEOREM 1.1

In the whole proof, we can work with Assumption (a') which does not restrict the generality as explained above. To avoid complicated formulas, we set

$$H(x) = 2(n-2)(n-4)\omega_{n-1}G_g(x, p).$$

By Relation (2), H satisfies the following expansion near p

$$H(x) = \frac{1}{d_g(x, p)^{n-4}} + B + \alpha(x) \quad (3)$$

where $B = 2(n-2)(n-4)\omega_{n-1}A$ and where $\alpha = 2(n-2)(n-4)\omega_{n-1}\alpha_p$ is smooth around p and satisfies $\alpha(p) = 0$. Theorem 1.1 is equivalent to show that $B \geq 0$ with equality if and only if (M, g) is conformally diffeomorphic to the standard sphere. For any metric g , let

$$L_g := \frac{4(n-1)}{n-2}\Delta_g + S_g$$

be the Yamabe operator. We recall some well known facts about L_g . The reader may refer to [Au98, He97] for further informations. First, as well as P_g , L_g is conformally covariant. If $g' = u^{\frac{4}{n-2}}g$ is conformal to g then

$$L_{g'}(u^{-1} \cdot) = u^{-\frac{n+2}{n-2}} L_g(\cdot) \quad (4)$$

It follows that the scalar curvatures S_g and $S_{g'}$ are related by the following equation

$$L_g u = S_{g'} u^{\frac{n+2}{n-2}}. \quad (5)$$

By Assumptions (a') and (b), the Green's function Λ_g of L_g exists, is unique, smooth and positive on $M \setminus \{p\}$. Setting $\Gamma(x) = 4(n-1)\omega_{n-1}\Lambda_g(x, p)$ to simplify formulas, we have when x is close to p

$$\Gamma(x) = \frac{1}{d_g(x, p)^{n-2}} + C + \beta(x) \quad (6)$$

where by $C \in \mathbb{R}$, β is a smooth function defined around p and satisfies $\beta(0) = 0$. We define a new metric $g' := \Gamma^{\frac{4}{n-2}}g$ conformal to g on $M_0 := M \setminus \{p\}$. Then, by (5)

$$S_{g'} = \Gamma^{-\frac{n+2}{n-2}}L_g(\Gamma) \equiv 0 \quad (7)$$

on M_0 . We set $H' = \Gamma^{-\frac{n-4}{n-2}}H$. By conformal covariance of the Paneitz-Branson operator (1) and since $P_g H = 0$ on M_0 , we have $P_{g'} H' \equiv 0$ on M_0 . Define for all $\epsilon > 0$ small enough, $M_\epsilon := M \setminus B^g(p, \epsilon)$ where $B^g(p, \epsilon)$ stands for the ball of center p and radius ϵ with respect to the metric g . We have

$$\int_{M_\epsilon} P_{g'} H' dv_{g'} = 0. \quad (8)$$

By Relation (7) and from the definition of P_g we have

$$P_{g'} H' = \Delta_{g'}^2 H' - \operatorname{div}_{g'} \left(\frac{4}{n-2} \operatorname{Ric}_{g'} dH' \right) - \frac{n-4}{(n-2)^2} |E_{g'}|^2 H'.$$

Set $S_\epsilon := \partial M_\epsilon = \partial B^g(p, \epsilon)$ be the $(n-1)$ -dimensional sphere of center p and radius ϵ . We let $ds_{g'}$ (resp. ds_g) be the volume element induced by g' (resp. g) on S_ϵ . Integrating by part the above relation, we obtain

$$\int_{M_\epsilon} P_{g'} H' dv_{g'} = -\mathbf{I} + \frac{4}{n-2} \mathbf{II} - \frac{1}{2} \int_{M_\epsilon} |E_{g'}|^2 H' dv_{g'} \quad (9)$$

where

$$\begin{cases} \mathbf{I} &= \int_{S_\epsilon} \partial_{\nu'} \Delta_{g'} H' ds_{g'} \\ \mathbf{II} &= \int_{S_\epsilon} \operatorname{Ric}_{g'}(\operatorname{grad}^{g'} H', \nu') ds_{g'}. \end{cases}$$

Here, ν' denotes the unit outer normal vector on $S_\epsilon = \partial M_\epsilon$ with respect to the metric g' .

2.1. Computation of I. First, we notice that the scalar curvatures S_g and $S_{g'}$ vanish on S_ϵ . For g , this comes from Assumption (a') and for g' , this follows from (7). Consequently, using Formula (4) and

$$\begin{aligned} \Delta_{g'} H' &= \frac{n-2}{4(n-1)} L_{g'} H' \\ &= \frac{n-2}{4(n-1)} \Gamma^{-\frac{n+2}{n-2}} L_g (\Gamma H') \\ &= \frac{n-2}{4(n-1)} \Gamma^{-\frac{n+2}{n-2}} L_g \left(\Gamma^{\frac{2}{n-2}} H \right) \end{aligned}$$

We obtain

$$\Delta_{g'} H' = \Gamma^{-\frac{n+2}{n-2}} \Delta_g \left(\Gamma^{\frac{2}{n-2}} H \right). \quad (10)$$

We set $r := d_g(x, p)$. From Formulas (3) and (6), we have:

$$\Gamma^{\frac{2}{n-2}} H = \left(\frac{1}{r^{n-2}} + C + \beta(x) \right)^{\frac{2}{n-2}} \left(\frac{1}{r^{n-4}} + B + \alpha(x) \right).$$

Then, using Taylor formula at p ,

$$\Gamma^{\frac{2}{n-2}} H = r^{2-n} + Br^{-2} + O(r^{-1})$$

where in the whole proof, $O(r^m)$ denotes a smooth function defined in a neighborhood of p and which satisfies

$$|\nabla_g^k O(r^m)|_g \leq C_k r^{m-k}$$

for all $k \in \mathbb{N}$. Since g is flat around p , we have for radially symmetric functions f ,

$$\Delta_g f(r) = -f''(r) - \frac{n-1}{r} f'(r). \quad (11)$$

Hence, this gives that near p ,

$$\Delta_g \Gamma^{\frac{2}{n-2}} H = 2(n-4)Br^{-4} + O(r^{-3})$$

and hence by (10) and (6)

$$\Delta_{g'} H' = \Gamma^{-\frac{n+2}{n-2}} \Delta_g H = 2(n-4)Br^{n-2} + O(r^{n-1}).$$

We then obtain

$$\frac{\partial}{\partial r} (\Delta_{g'} H') = 2(n-2)(n-4)Br^{n-3} + O(r^{n-2}). \quad (12)$$

On S_ϵ , $r \equiv \epsilon$. In addition,

$$\nu' = -\Gamma^{-\frac{2}{n-2}} \frac{\partial}{\partial r} = -(\epsilon^2 + o(\epsilon^2)) \frac{\partial}{\partial r} \quad (13)$$

and

$$ds_{g'} = \Gamma^{\frac{2}{n-2}} ds_g = \Gamma^{\frac{2}{n-2}} \epsilon^{n-1} ds = (\epsilon^{1-n} + o(\epsilon^{1-n})) ds. \quad (14)$$

where ds stands for the standard volume element on the unit $(n-1)$ -sphere. By Formulas (12), (13) and (14), we obtain

$$\mathbf{I} = -2(n-2)(n-4)\omega_{n-1}B + o(1) \quad (15)$$

2.2. Computation of II. If $g' = e^{2f}g$ is conformal to g , then the following formula holds (see [He97] p. 240 or [Au98]):

$$\text{Ric}_{g'} = \text{Ric}_g - (n-2)\nabla^2 f + (n-2)\nabla f \otimes \nabla f + (\Delta_g f - (n-2)|\nabla f|_g^2)g. \quad (16)$$

In this context, $f = \frac{2}{n-2} \log(\Gamma)$. By (6), we have near p

$$\begin{aligned} f &= \frac{2}{n-2} \log \left(\frac{1}{r^{n-2}} + O(1) \right) \\ &= -2 \log(r) + O(r^{n-2}). \end{aligned} \quad (17)$$

Let $(r, \Theta_1, \dots, \Theta_{n-1})$ be polar coordinates on \mathbb{R}^n . The Christoffel symbols Γ_{r, Θ_i}^r of the Euclidean metric in these coordinates identically vanish. This implies that for any radially symmetric function h , the mixed terms $\nabla_{r, \Theta_i}^2 h$ are zero. Since g is flat near p , we deduce that

$$\nabla^2 f = \frac{2}{r^2} dr^2 + b + \bar{O}(r^{n-4})$$

where, as in what follows, we denote by $\bar{O}(r^m)$ a 2-form whose norm with respect to g is $O(r^m)$ and where b is a 2-form such that

$$b \left(\cdot, \frac{\partial}{\partial r} \right) \equiv 0. \quad (18)$$

Using (11), one also computes that

$$\begin{aligned}\nabla f \otimes \nabla f &= \frac{4}{r^2} dr^2 + \bar{O}(r^{n-4}) \\ \Delta_g f &= \frac{2(n-2)}{r^2} + O(r^{n-4}) \\ |\nabla f|_g^2 &= \frac{4}{r^2} + O(r^{n-4}).\end{aligned}$$

Since g is flat near p , Ric_g vanishes and $g = dr^2 + r^2 \sigma^{n-1}$ where σ^{n-1} stands for the usual metric on the standard sphere \mathbb{S}^{n-1} . We deduce from these computations that

$$\begin{aligned}\text{Ric}_{g'} &= -(n-2)b - \frac{2(n-2)}{r^2} dr^2 + \frac{4(n-2)}{r^2} dr^2 + \\ &\quad \left(\frac{2(n-2)}{r^2} - \frac{4(n-2)}{r^2} + O(r^{n-4}) \right) (dr^2 + r^2 \sigma^{n-1}) + \bar{O}(r^{n-4}) \\ &= -(n-2)b - 2(n-2)\sigma^{n-1} + \bar{O}(r^{n-4}).\end{aligned}$$

We get from (3), (6) and the definition of H' that on S_ϵ

$$\begin{aligned}\text{grad}^{g'} H' &= \Gamma^{-\frac{4}{n-2}} \text{grad}^g (1 + O(r^{n-4})) \\ &= O(r^{n-1}) \frac{\partial}{\partial r} + v\end{aligned}\tag{19}$$

is a vector field such that $\text{Ric}_{g'}(v, \nu') = 0$. Observe that by (13) and (18), we have $\sigma^{n-1}(\cdot, \nu') = 0$ and $b(\cdot, \nu') = 0$ on S_ϵ . In addition, the estimates (13), (18) then imply that on S_ϵ

$$\begin{aligned}\text{Ric}_{g'}(\text{grad}^{g'} H', \nu') &= \bar{O}(r^{n-4})(\text{grad}^{g'} H', \nu') \\ &= O(\epsilon^{2n-3}).\end{aligned}$$

Relation (14) then leads to

$$\mathbf{II} = O(\epsilon^{n-2}) = o(1).\tag{20}$$

2.3. Conclusion. Using (8), (9), (15), (20) and passing to the limit $\epsilon \rightarrow 0$, we obtain that

$$0 = 2(n-2)(n-4)\omega_{n-1}B - \frac{1}{2} \int_{M \setminus \{p\}} |E_{g'}|^2 H' dv_{g'}.\tag{21}$$

Assumption **(e)** implies that $H' > 0$ and hence $B \geq 0$. This proves first part of Theorem 1.1.

Now, assume that $B = 0$. Then $E_{g'} \equiv 0$ on $M \setminus \{p\}$. This implies that $(M \setminus \{p\}, g')$ is Einstein and scalar flat hence Ricci flat. Since in addition the Weyl curvature is zero, $(M \setminus \{p\}, g')$ turns to be flat (see [He97] p. 123). It is known that $(M \setminus \{p\}, g')$ is asymptotically flat and that its mass satisfies $m(g') = c_n C$ where $c_n > 0$ (see e.g. Lee-Parker [LP87]). Since g' is flat, $m(g') = 0$ so is C and by a positive mass Theorem by Schoen-Yau [SY88], (M, g) is conformally diffeomorphic to (S^n, g) .

Remark 2.1. It is clear from the proof that Assumption **(a)** can be weakened and replaced by

(a) g is locally flat around a point p and the standard Positive Mass Theorem is valid on M (i.e. with the notations of Section 2, $C \geq 0$ with equality if and only

if (M, g) is conformally diffeomorphic to \mathbb{S}^n). In particular, by [SY79] and [AH03], this assumption holds if $n \in \{5, 6, 7\}$ or if M is spin.

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